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A Survey of Knot Concordance

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Low Dimensional Topology of Tomorrow

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At the opening of the conference, our host, Hitoshi Murakami, charged the participants to consider the past, present, and future of low-dimensional topology. I therefore arrange this paper to follow that format.

1 The Past

1.1 Knot Concordance

A *knot* is an embedding of an oriented S^1 in an oriented S^3. We work here up to diffeomorphism or piecewise-linear locally flat homomorphism. A knot K is concordant to a knot J ($K \simeq J$) if there exists a manifold pair W, consisting of an annulus embedded in a three sphere cross an interval, $W \cong (S^1 \times I) \hookrightarrow (S^3 \times I)$, such that $\partial W \cong K \cup -J$. Here, $-J$ means the knot J with the orientation reversed on both the ambient S^3 and the embedded S^1.

Concordance is an equivalence relation, and it respects connected sums of knots; i.e. if $K \simeq K'$ and $J \simeq J'$ then $K\#J \simeq K'\#J'$. Thus we can form a group $\mathcal{C}_1$ out of knots, under the operation of connected sum. The identity of the group is the unknot. To see that this group contains inverses, and to discuss other knots that are in the same equivalence class as the unknot, we next consider slice knots.

1.2 Slice Knots

Slice knots were first defined by Fox and Milnor [3]. They defined slice knots as the knots obtained from taking a 2-sphere embedded in a 4-sphere, and intersecting with a standardly embedded 3-sphere. The intersection of a 2-sphere with a 3-sphere is generically a collection of 1-spheres; if there is only one component in this collection, then that 1-sphere in the standardly embedded 3-sphere is a slice knot. Fox and Milnor asked what kind of knots can occur as such slices. A more recent definition is that a knot K is slice if it bounds a two-disk embedded in a four-ball, i.e. $K = S^1 \hookrightarrow S^3 = \partial(D^2 \hookrightarrow B^4)^*$. Fox's and Milnor's question has been turned around to ask, given a knot, is it slice?

The connection of sliceness to concordance is that K is slice if and only if K is the identity in $\mathcal{C}_1$. With a bit of traditional three-manifold cut-and-paste and one can see that $K\# -J$ is slice if and only if $K \simeq J$. This gives us our inverses in $\mathcal{C}_1$; since clearly $K \simeq K$, we have $K\# -K$ is slice, and so every knot has an inverse in $\mathcal{C}_1$. Also, we can now think of $\mathcal{C}_1$ as knots modulo slice knots.

*I use two different notations for an n-disk to provide a convenient vocabulary: we can say "slice disk" and "bounding ball" and the meaning of each is clear.

1.3 A Necessary Condition for K To Be Slice

The first* obstruction to a knot being slice is found in the Seifert form of a knot. Let K be a slice knot with slice disk D and Seifert surface F. $F \cup D$ bounds a 3-manifold M in the bounding ball. Easy algebraic topology gives that, for the inclusion map $i \colon F \hookrightarrow M$, $\mathrm{Ker}\big(H_1(F) \xrightarrow{i_*} H_1(M)\big)$ has half the rank of $H_1(F)$. A geometric argument shows that for $a, b \in \mathrm{Ker}\big(H_1(F) \xrightarrow{i_*} H_1(M)\big)$, the Seifert form V vanishes on a and b, i.e. $V(a,b) = 0$. To see why, first note that since a and b are in the kernel of i_*, there are surfaces $A, B \subset M$ such that $a = \partial A, b = \partial B$. Recall that the Seifert form $V(a,b)$ is the linking number of a "pushed off" of F and b; i.e. $V(a,b) = lk(a^+, b)$. The linking form in S^3 is the same as intersection in B^4, thus $lk(a,b) = \mathrm{int}(A^+, B)$. Because A^+ is "pushed" out of M, $A^+ \cap M = \emptyset$, so $\mathrm{int}(A^+, B) = 0$. To summarize:

Proposition. *K slice implies that there exists $H \subset H_1(F)$, with rank $H = 1/2$ rank $H_1(F)$, such that $V|_{H \times H} \equiv 0$.*

1.4 Algebraic Sliceness

It was not known at first if the above implication is true in the other direction. To have the terminology with which to attack that question, we take the converse of the theorem and make from it a definition. Thus:

Definition. A knot K is algebraically slice if for F a Seifert surface of K, there exists

*either ordered historically or by decreasing simplicity

$H \subset H_1(F)$, rank $H = 1/2$ rank $H_1(F), V|_{H\times H} \equiv 0$.

That is, in some basis V can be represented as a matrix

$$V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

with A, B and C matrices of half the size of V. Forms which have a half-size* sub-form which is zero is a theme throughout this theory; the half-size sub-object on which such a form disappears is called a *metabolizer*, the form is called *metabolic.*

Note that since the Alexander polynomial can be found as $\det(V - tV^t)$, if K is algebraically slice, then its Alexander polynomial factors as $\Delta_K(t) = f(t)f(t^{-1})$.

As an aid to finding the structure of the concordance group $\mathcal{C}_1$, we define the algebraic concordance group $\mathcal{G}$ as Seifert forms modulo algebraically slice Seifert forms. Then we have an onto map $\mathcal{C}_1 \to \mathcal{G}$.

Levine investigated $\mathcal{G}$, and found a set of invariants that fully classified it [9]. In fact, he showed that $\mathcal{G} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty$. He also investigated higher-dimensional analogues of $\mathcal{G}$ and $\mathcal{C}_1$, and found that $\mathcal{G} \cong \mathcal{C}_n$ for $n \geq 2$†

In the classical knot dimension, the question "Are there algebraically slice knots that are not slice?" remained open. The question can now be restated as, "Is there a non-trivial kernel of $\mathcal{C}_1 \to \mathcal{G}$?"

*size meaning different things in different contexts

†Those who do low-dimensional topology should recognize the pattern; again, higher dimensional topological questions are completely determined by the algebra.

2 The Past and the Present

2.1 Casson - Gordon Invariants

Casson and Gordon [1] created an invariant that gave a positive answer to the above question. Let K be a knot, n and d powers of primes, and denote by K_n ($\bar{K}_n$) the n-fold (branched) cyclic cover of the knot complement. Choose a map $\chi\colon H_1(\bar{K}_n) \to \mathbb{Z}/d$; the composition $\pi_1(K_n) \to H_1(K_n) \xrightarrow{i_*} H_1(\bar{K}_n) \xrightarrow{\chi} \mathbb{Z}/d$ gives you a d-fold cyclic cover of K_n. The manifold K_n already has an infinite cyclic cover which it shares with K, so we get a diagram of covers:

$$\begin{tikzcd} K_\infty \arrow[d] & K_{\infty,d} \arrow[l] \arrow[d] \\ K_n \arrow[d] & K_{n,d} \arrow[l] \\ K & \end{tikzcd}$$

Do 0-surgery on the lifts of the knot in the manifolds K_n, K_∞, ,$K_{n,d}$, $K_{\infty,d}$ to create $\hat{K}_n$, $\hat{K}_\infty$, ,$\hat{K}_{n,d}$, $\hat{K}_{\infty,d}$; we do this in order to have manifolds without boundary, but with the same fundamental groups as the originals. This allows us to appeal to bordism theory and find that $\Omega_3(\mathbb{Z} \times \mathbb{Z}/d)$ is finite, so there exist a set of four-manifolds and an integer r such that

$$r \cdot \begin{tikzcd} \hat{K}_\infty \arrow[d] & \hat{K}_{\infty,d} \arrow[l] \arrow[d] \\ \hat{K}_n & \hat{K}_{n,d} \arrow[l] \end{tikzcd} = \partial \begin{tikzcd} V^4_\infty \arrow[d] & V^4_{\infty,d} \arrow[l] \arrow[d] \\ V^4_n & V^4_{n,d} \arrow[l] \end{tikzcd}$$

The $\mathbb{Z} \times \mathbb{Z}/d$ cover of V_n gives us a twisted homology $H_*(V_n; \mathbb{Q}(\mathbb{Z} \times \mathbb{Z}/d))$. This is homology on a four-manifold; according to Wall [18] we get an intersection form on the second homology $H_2(V_n; \mathbb{Q}(\mathbb{Z} \times \mathbb{Z}/d))$. Call this intersection form $t(V)$; it is the most important ingredient in the Casson-Gordon τ invariant:

$$\tau(K, \chi) = \frac{1}{r}\left(t(V_n) - t_0(V_n)\right).$$

The $t_0(V_n)$ is the intersection form on V_n in ordinary, untwisted homology $H_2(V_n; \mathbb{Q})$; it and r are needed in the definition to be sure that the invariant is well-defined, and does not depend on the choice of the four-manifold V_n. The invariant lives in a Witt group tensored with the rational numbers. A Witt group is a group of Hermitian bilinear forms over a given field, modulo metabolic forms. Recall that metabolic forms were to be a theme here: just as a knot is algebraically slice if its Seifert from is metabolic, its Casson-Gordon invariant is trivial if the form τ is metabolic. The Witt group is tensored with the rational numbers to allow multiplication by $1/r$.

Of course, this invariant is introduced in this paper because it detects slice knots:

Theorem (Casson, Gordon [1]). *Let the knot K have n-fold branched cyclic cover $\bar{K}_n$, with n a power of a prime. If K is slice, then there is a subgroup H of $H_1(\bar{K}_n)$ such that the intersection pairing on H is identically zero, and $\tau(K, \chi) = 0$ for every $\chi\colon H_1(\bar{K}_n) \to \mathbb{Z}/d$, d a prime power, $\chi(H) = 0$.*

No exposition of the details of this intricate proof is better than the original. Any summary or sketch can not do justice to the proof, but basically, if a knot is slice, it is shown that one can use covers of B^4 branched over D^2 for the four-manifolds V_*.

2.2 Logic

Casson's and Gordon's theorem, rewritten with symbols for universal and existential quantifiers, and skipping some qualifying phrases, reads

Theorem (Casson, Gordon). *K slice* $\Longrightarrow$

$$\exists H \subset H_1(\bar{K}_n)\ \forall \chi\colon H_1(\bar{K}_n) \to \mathbb{Z}/d, \tau(K,\chi) = 0.$$

Clearly this theorem can only be used to prove that some knot is not slice, by using the contrapositive:

Theorem (Casson, Gordon). $\forall H \subset H_1(\bar{K}_n)\ \exists \chi\colon H_1(\bar{K}_n) \to \mathbb{Z}/d, \tau(K,\chi) \neq 0 \implies$ *K not slice.*

The basic tactic is this: list all subgroups H of $H_1(\bar{K}_n)$, and find a χ for each such that the invariant does not disappear. It is therefore important to restrict the subgroups H which one must consider. There are three basic restrictions: the size of the group H is restricted to $|H| = \sqrt{|H_1(\bar{K}_n)|}$; the linking form must be trivial on H, i.e. $\mathrm{link}|_{H\times H} \equiv 0$; and H is invariant under deck transformations. Hopefully this serves to limit the possible Hs to two or three; one then finds an appropriate χ for each.

The problem of actually calculating τ is less easily surmountable.

2.3 Twisted doubles of the unknot

Casson and Gordon were the first to use this invariant; they used it to find knots that are algebraically slice but not slice; thus finding non-trivial elements of the kernel of $\mathcal{C}_1 \to \mathcal{G}$. The knots in question are twisted doubles of the unknot.

A twisted double of the unknot is algebraically slice if the number of twists equals $n(n-1)$ for some n. These knots were chosen because their two-fold covers are lens spaces, which are well understood. Particularly useful are the facts that, first, if the twisted double of an unknot is algebraically slice, the first homology of the branched two-fold cover is $\mathbb{Z}/k$, k a square. Second, Casson and Gordon could use Atiyah-Singer index theory to find expression for signature of intersection form on $H_2(V_n; \mathbb{Q}(\mathbb{Z}_d))$. They then proved that this signature can be used to approximate the signature of τ. This sufficed to prove that all the invariants τ were non-trivial.

If the number of twists in the twisted double of the unknot is 0, then the knot is the unknot, which is slice. If the number of twists is 2, then the knot is the stevedore's knot, which is slice. Casson and Gordon proved that for an algebraically slice twisted double of the unknot, for any number of twists greater than two, the knot is not slice. Thus $\mathrm{Ker}(\mathcal{C}_1 \to \mathcal{G}) \neq \{0\}$, and the kernel in fact contains an infinite family of knots.

2.4 A formula for τ

Gilmer [4] made two great steps in the use of the Casson-Gordon invariant. Firstly, he related maps $H_1(\bar{K}_n;\mathbb{Z}) \to \mathbb{Z}/d$ that disappear on appropriate subgroups $H \subset H_1(\bar{K}_n)$ to elements in specific subgroups of the homology of the Seifert surface of the knot itself. This made finding appropriate Hs and χs for Casson and Gordon's theorem simpler, as they could be sought in the homology of the Seifert surface, rather than in homologies of cyclic covers. Secondly, in certain cases, he provided a formula with which one could calculate τ.

Theorem (Gilmer [4]). *For K an algebraically slice genus 1 knot, for a 2-fold cover,*

$$\tau(K,\chi) = \rho\left(2\sigma_{\frac{s}{d}}(J_x) + \frac{4(d-s)s}{d^2}V(x,x) - \sigma_{\frac{1}{2}}(K)\right).$$

Let F be a Seifert surface for the knot, and V the Seifert form. In the theorem, $x \in H_1(F)$ such that $d|V(x,x)$, J_x is a knot on F representing x, σ_α is the Tristram-Levine signature, and ρ is a partial inverse to the signature map on the Witt group where Casson-Gordon invariants live.

Gilmer gave an example of a knot K, reproduced in Figure 1, that is algebraically slice and not slice. A Seifert surface can be seen in the diagram. The twists in the bands of the Seifert surface are there to make the surface have a specific Seifert form, in this

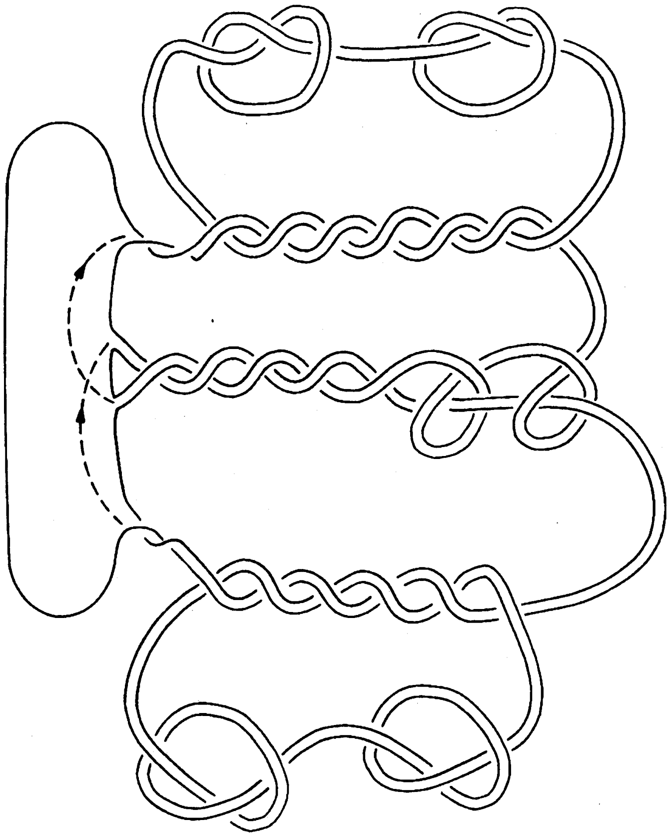

Figure 1:

case that form is

$$\begin{pmatrix} 0 & 4 \\ 5 & 0 \end{pmatrix},$$

in the basis indicated by the dashed lines. The Seifert form tells us that the knot is algebraically slice. The knot cover has homology $H_1(\bar{K}_2) \cong \mathbb{Z}/9 \oplus \mathbb{Z}/9$. The curve J_x is two copies of the trefoil knot connect summed with the mirror image of the $(2,7)$ torus knot. The subgroup H in Casson's and Gordon's theorem needs to have order 9, thus such a subgroup can be isomorphic to $\mathbb{Z}/9$ or to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$. Using theorems relating these Hs to subgroups of $H_1(F)$, all candidates for H can be eliminated, except for two, each homomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/3$. With clever choices for χs, and the formula above, τ was calculated for each of these Hs, and shown non-trivial for each one.

2.4.1 Another formula for τ

Naik [13] extended Gilmer's formula for τ. By creating a specific four-manifold that could be used for the V in finding the Casson-Gordon invariant, she could write a more general formula:

Theorem (Naik [13]). *For a genus 1 knot K, and an n-fold cover, where x a generator of $H_1(F)$, J_x a curve representing x, and $d|V(x,x)$,*

$$\tau(K,\chi) = \sum_{i=0}^{n-1}\left(\sigma_{\frac{s_i}{d}}(J_x) + \frac{2(d-s_i)s_i}{d^2}V(x,x) - \sigma_{\frac{i}{n}}(K)\right)$$

Without going into details, the s_i are mod-d integers related to $V(x,y)$.

Trotter [16] was the first to show that there are knots that are not equal to their reverses. A reverse of a knot is the knot with the orientation of the S^1 reversed. We will use the notation K^r to denote the reverse of K, and K^m to denote the mirror of K; the mirror of a knot is the knot with the orientation reversed on the ambient S^3. Note that $-K = K^{rm}$. Trotter's examples of knots not equal to their reverses were pretzel knots.

Reverses and mirrors of pretzel knots can be summarized as:

$$\begin{tikzcd}
\text{pretzel}(p,q,r) \arrow[r, "\text{reverse}"] \arrow[d, "\text{mirror}"'] & \text{pretzel}(r,q,p) \arrow[d, "\text{mirror}"'] \\
\text{pretzel}(-p,-q,-r) \arrow[r, "\text{reverse}"] & \text{pretzel}(-r,-q,-p)
\end{tikzcd}$$

Naik used the formula to show that two of Trotter's examples, the $(3,-5,7)$ pretzel knot and the $(3,-5,17)$ pretzel knot, are not concordant to their reverses. To do this in the $(3,-5,7)$ case, one needs to consider whether the knot

$$\text{pretzel}(3,-5,7)\# -\text{pretzel}(3,-5,7)^r$$

is slice; i.e. whether

$$\text{pretzel}(3,-5,7)\#\,\text{pretzel}(3,-5,7)^m = \text{pretzel}(3,-5,7)\#\,\text{pretzel}(-3,5,-7)$$

is slice. Pretzel knots are not, in general, algebraically slice, but they are genus one. The sum of the knots pretzel$(3,-5,7)\#$ pretzel$(-3,5,-7)$ is algebraically slice. By using the above formula, and the fact from Gilmer [4] that $\tau(K\#J, \chi_K \oplus \chi_j) = \tau(K,\chi_K) + \tau(J,\chi_J)$, the calculation of Casson-Gordon invariants is possible. The fact that these invariants vanish proves that the two pretzel knots $(3,-5,7)$ and $(3,-5,17)$ are not concordant to their reverses.

2.5 Twisted Alexander Polynomial

Despite great advances in calculating Casson-Gordon invariants, these invariants are still not easy to find in general. One invariant that is calculable in general, and is related to the Casson-Gordon invariant, is the twisted Alexander polynomial. The following definition of the twisted Alexander polynomial follows Kirk and Livingston [7] in the specific case that applies to sliceness of knots.

As in the setup for the Casson-Gordon invariant, n and d be powers of primes, let K be a knot, let K_n be the n-fold cyclic cover, and let $\bar{K}_n$ be the n-fold branched cyclic cover. Again take $\chi\colon H_1(\bar{K}_n) \to \mathbb{Z}/d$. Then we have a map $\epsilon\colon \pi_1(K_n) \to \langle t \rangle$ and a map $\rho\colon \pi_1(K_n) \to Aut(\mathbb{Q}(\zeta))$, where ζ is a primitive dth root of unity. The map ρ is defined by the composition

$$\pi_1(K_n) \xrightarrow{inc_*} \pi_1(\bar{K}_n) \to H_1(\bar{K}_n) \xrightarrow{\chi} \mathbb{Z}/d \to Aut(\mathbb{Q}(\zeta)),$$

where an element of $\mathbb{Z}/d$ acts on $\mathbb{Q}(\zeta)$ by $a \cdot z = \zeta^a z$.

The group $\pi_1(K_n)$ acts on $C_*(\tilde{K})$, the chains of the universal cover of K; $\pi_1(K_n)$ also acts on $\mathbb{Q}(\zeta)[t, t^{-1}]$ via $\epsilon \otimes \rho$, the action given by

$$\gamma \cdot \sum z_i t^i = t^{\epsilon(\gamma)} \sum \rho(\gamma)(z_i) t^i.$$

Thus we can form the tensor product

$$C_*(\tilde{K}) \otimes_{\pi_1(K_n)} \mathbb{Q}(\zeta)[t,t^{-1}].$$

We take the homology of that chain complex and call it $H_*(K;\mathbb{Q}(\zeta)[t,t^{-1}])$. This is a twisted homology of K_n, twisted by the $d \times \infty$ cover we get from the maps ρ and ϵ.

Since $\mathbb{Q}(\zeta)[t,t^{-1}]$ is a PID, we can write

$$H_i(X;\mathbb{Q}(\zeta)[t,t^{-1}]) \cong \frac{\mathbb{Q}(\zeta)[t,t^{-1}]}{p_1(t)} \oplus \frac{\mathbb{Q}(\zeta)[t,t^{-1}]}{p_2(t)} \oplus \ldots \oplus \frac{\mathbb{Q}(\zeta)[t,t^{-1}]}{p_n(t)} \oplus (Q(\zeta)[t,t^{-1}])^N.$$

The twisted Alexander polynomial is then defined as the order of the torsion of the twisted homology $H_i(X;\mathbb{Q}(\zeta)[t,t^{-1}])$, i.e.

$$\Delta_{i,\rho}(t) = \prod p_k(t).$$

Since this polynomial depends only on K and χ, we usually write it as $\Delta_\chi(t)$. Two important theorems regarding these polynomials are:

Theorem (Kirk, Livingston [7]). *Let $\pi_1(K_n) = \langle g_1,\ldots,g_n | r_1,\ldots,r_{n-1}\rangle$. Form the Jacobian of Fox derivatives $(\partial r_i/\partial g_j)$ and let M be the matrix obtained from applying $\epsilon \otimes \rho$ to the elements of the Jacobian. Let g_k be a generator such that $\epsilon \otimes \rho(g_k)$ is non-*

trivial; let M_k be the matrix M with the kth column removed. Then

$$\Delta_\chi(K) = \frac{\det(M_k)}{1 - \epsilon \otimes \rho(g_k)} \gcd(1 - \epsilon \otimes \rho(g_i)).$$

The definition of a twisted Alexander polynomial as $\det(M_k)/(1 - \epsilon \otimes \rho(g_k))$ was given by Wada [17]; the theorem in [7] proves that the two definitions are related.

The theorem that relates the twisted Alexander polynomial to Casson-Gordon invariants is

Theorem (Kirk, Livingston [7]). *Up to norms of polynomials and invertible elements of $\mathbb{Q}(\zeta)[t, t^{-1}]$,*

$$\Delta_\chi(K)(1-t)^e = \det(\tau(K, \chi)).$$

Here e is either 0 or 1, depending on whether ρ is non-trivial or not.

The first theorem shows how to calculate $\Delta_\chi(t)$ in a purely algorithmic way; the second connects it to the Casson-Gordon invariant and thus to the sliceness of knots. Specifically, the second theorem implies that if K is slice, then $\Delta_\chi(t)$ factors as $f(t)\bar{f}(t^{-1})(1-t)^e$, where $\bar{f}(t)$ is the complex conjugate of $f(t)$. These theorems were used by Kirk and Livingston [8] to show that the knot 8_{17} is not concordant to its reverse.

2.6 Order 2 in $\mathcal{G}$

As an application of the twisted Alexander polynomial, we consider the question of whether knots of order 2 in $\mathcal{G}$ are order 2 in $\mathcal{C}_1$. As we know, there is a direct summand of $\mathcal{G}$ which is the infinite direct sum of copies of $\mathbb{Z}/2$; it turns out that the pre-image of $(\mathbb{Z}/2)^\infty$ in $\mathcal{C}_1$ is not all of order two.

Some examples of knots that of order two in $\mathcal{G}$ are: k-twisted doubles of the unknot for $4k+1$ prime, $k \geq 3$, as well as 8_{13}, 9_{14}, 9_{19}, 9_{30}, 9_{33}, 9_{44}, 10_{10}, 10_{13}, 10_{26}, 10_{28}, 10_{34}, 10_{58}, 10_{60}, 10_{91}, 10_{102}, 10_{119}, 10_{135}, 10_{158}, and 10_{165} from Rolfsen's [14] table. It has been shown [15] that none (except 10_{158}) of the above are order two in $\mathcal{C}_1$; 8_1 and 10_1 are, in fact, of infinite order in $\mathcal{C}_1$. Two different methods were used. For all the knots K except 8_1 and 10_1, twisted Alexander polynomials were computed for $K\#K$. This, of course, required investigation of subgroups H of $H_1(\bar{K}_n)$, and finding appropriate χs for each one, so that the twisted Alexander polynomials do not factor as the theorem requires, proving the knots not slice. The knots 8_1 and 10_1 are twisted doubles of the unknot; these were shown to have infinite order in $\mathcal{C}_1$ via methods similar to Casson's and Gordon's [1].

2.7 Order 4 in $\mathcal{G}$

Some more general results have been found regarding elements of order 4 in $\mathcal{G}$. According to Levine [9], if a knot has Alexander polynomial $\Delta_K(t) = at^2 - (2a+1)t + a$, and if further $|H_1(\bar{K}_2)| = \Delta_K(-1) = p^n m$, with p a prime such that $p \equiv 3 \pmod 4$, n odd, $\gcd(p, m) = 1$, then the knot K is of order 4 in $\mathcal{G}$. There are many examples of such knots: many twisted doubles of the unknot are such, as are many knots with eleven or fewer crossings on standard knot tables. As a contrast, many such knots are of infinite order in $\mathcal{C}_1$:

Theorem (Livingston, Naik [12]). *For K a knot, $H_1(\bar{K}_2) \cong \mathbb{Z}_{p^n} \oplus G$, p a prime such that $p \equiv 3 \pmod 4$, n odd, $p \nmid |G|$, then K is infinite order in $\mathcal{C}_1$.*

The paper defines characters $H_1(\bar{K}_n) \to \mathbb{Z}/p^n$ via the linking form. Choose an element $a \in H_1(\bar{K}_n)$, and define the character χ_a as $\operatorname{link}(a, \cdot)$. Also, rather than the Casson-Gordon τ invariant, the invariant σ is used, which is a certain signature of τ.

Here, the method of proof is not the usual of finding all metabolizers and then calculating Casson-Gordon invariants, but rather a logically related tactic of contradiction. Consider a knot K that fulfills the hypothesis of the theorem, but which has finite order in $\mathcal{C}_1$. According to Levine, the order of K in $\mathcal{G}$ is 4, so if K is to be of finite order in $\mathcal{C}_1$, then that order is $4k$. The p-primary component of the metabolizer in $H_1((4k)\bar{K}_2)$ from Casson's and Gordon's theorem is isomorphic to $(\mathbb{Z}_{p^n})^{2k}$, generated by vectors $v_1, \ldots, v_{2k}$.

By a change of basis, we can assume that these vectors are:

$$(1, 0, \ldots)$$

$$(0, p, 0, 0, 0, \ldots)$$

$$(0, 0, p, 0, 0, \ldots)$$

$$(0, 0, 0, p, 0, \ldots)$$

$$(0, 0, 0, 0, p^2, 0, 0, 0, \ldots)$$

$$(0, 0, 0, 0, 0, p^2, 0, 0, \ldots)$$

$$(0, 0, 0, 0, 0, 0, p^2, 0, \ldots)$$

$$\vdots$$

With some linear algebra work, we can create a vector $(p^{n-1}, \ldots, p^{n-1}, *, \ldots, *)$ in the metabolizer. Since K is slice, $\sigma((4k)K, \chi_{(p^{n-1}, \ldots, *)}) = 0$. It is then shown that this implies $\sigma(K, \chi_{p^{n-1}}) = 0$. Then an induction argument and further manipulation of the vectors listed above show $\sigma(K, \chi_{p^{(n-1)/2}}) = 0$; this is a contradiction to the fact that

$$\sigma(K, \chi_{p^{(n-1)/2}}) = \text{link}(p^{(n-1)/2}, p^{(n-1)/2}) \mod \mathbb{Z},$$

and that the linking form is non-singular.

As a result, we have the following corollaries:

Corollary (Livingston, Naik [12]). *Let K be a knot K with Alexander polynomial*

$\Delta(t) = nt^2 - (2n+1)t + n$, *with* $4n + 1 = p^{odd}q$, $p \equiv 4 \pmod 4$, *with the p-primary summand of* $H_1(\bar{K}_2)$ *cyclic. Then* K *is of infinite order in* $\mathcal{C}_1$.

Corollary (Livingston, Naik [12]). *A two-bridge knot* $K_{m/n}$ *is infinite order in* $\mathcal{C}_1$ *if some prime congruent to 3 mod 4 has odd exponent in* m.

Corollary (Livingston, Naik [12]). *Any twisted double of a knot is order 4 in* $\mathcal{G}$ *is of infinite order in* $\mathcal{C}_1$.

2.8 The Alexander Polynomial, and splitting τ

S. Kim [5] found a connection between the Alexander polynomial and Casson-Gordon invariants.

Theorem (S. Kim [5]). *Let* K_1, K_2 *be knots such that their Alexander polynomials have no common factors, and so that* K_1 *has a non-singular Seifert form. Then if Casson-Gordon invariant of* $K_1 \# K_2$ *is zero, so are the invariants of* K_1 *and* K_2.

This theorem makes it easier to find knots that are linearly independent in $\mathcal{C}_1$; all that is necessary is that they be infinite order in $\mathcal{C}_1$, and that their (untwisted) Alexander polynomials have no common factors.

Also proved was:

Theorem (S. Kim [5]). *All but finitely many twisted doubles of a knot are of infinite order in* $\mathcal{C}_1$.

As a corollary to these two theorems, since the Alexander polynomial of a twisted double of a knot is $kt^2 - (2k+1)t + k$, where k is the number of twists, we have that all but finitely many twisted doubles of a knot are linearly independent in $\mathcal{C}_1$. Thus there is a copy of $\mathbb{Z}^\infty$ in $\mathcal{C}_1$.

2.9 Order 2 in Ker($\mathcal{C}_1 \to \mathcal{G}$)

More work on finding subgroups of $\mathcal{C}_1$ was done by Livingston [11]. He created a family of knots that represents an infinite direct sum of copies of $\mathbb{Z}/2$ in the kernel of $\mathcal{C}_1 \to \mathcal{G}$. Let T be a knot, let K_T be the knot created by tying the knots T and $-T$ in the bands of a knot, as shown in Figure 2. Note that since $K_T = -K_T$, K_T is order 2 in $\mathcal{C}_1$.

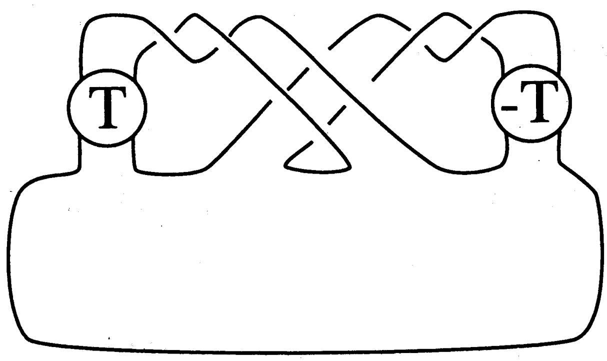

Figure 2:

As in Gilmer's example above, the twists in the bands determine the algebraic concordance class of K_T; we arrange to have K_T have a Seifert form

$$V = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Litherland [10] has formulas for calculating the Casson-Gordon invariants of satellite knots. Using these formulas, one finds

$$\sigma(K_T, \chi_a) = \sigma(K_0, \chi) + 2\sigma_{\frac{a}{5}}(T) + 2\sigma_{\frac{3a}{5}}(-T).$$

The knot K_0 is the knot as described above, but with T the unknot.

Now let $J_i = K_0 \# K_{\#_i \,\mathrm{torus}(2,7)}$. With the twists in the bands correctly arranged, the knot K_0 and $K_{\#_i \,\mathrm{torus}(2,7)}$ represent the same class in $\mathcal{G}$; and since these knots are of order two, J_i algebraically slice. Further, for $i \neq j$, the knots J_i and J_j can be proven to be not concordant; thus we have that $\mathrm{Ker}(\mathcal{C}_1 \to \mathcal{G})$ contains a subgroup isomorphic to $\mathbb{Z}_2^\infty$.

2.10 Beyond the Casson-Gordon invariant

Of course, having a vanishing Casson-Gordon invariant does not guarantee that a knot is slice, and efforts were made to extend Casson's and Gordon's work to find a finer invariant. Cochran, Orr, and Teichner [2] found an infinite filtration of $\mathcal{C}_1$. The first step of this filtration implies algebraic sliceness, and the second implies a vanishing Casson-Gordon

The idea is to try to construct slice disk from a *grope*. Take a knot K in S^3, with D^4 bounding S^3. Embed a once-punctured oriented surface, possibly with genus, in D^4, with the boundary identified with K. If the surface has genus, find curves on the surface representing generators of the homology of the surface, and attach oriented, punctured surfaces to each of these. If, at any level, the surfaces at that level have no genus, then one can do surgery along all the surfaces in the grope and create a slice disk for the knot.

Cochrain, Orr, and Teichner defined (n) solvability of a knot, and showed that a knot is (n)-solvable if it bounds a grope of height $n+2$. More specifically, they defined (n) solvability and $(n.5)$ solvability for knots; (n) solvability implies the existence of a certain obstruction, and $(n.5)$ solvability implies that that obstruction vanishes. The definition carries on the theme of metabolizers and metabolic forms. A knot is (n) solvable if there exists a sequence of n submodules P_n of Alexander modules $\mathcal{A}_n$, the choice of each P_n dependent on the choice of all the previous ones. Each of the submodules P_n is a metabolizer for a generalized Blanchfield pairing. Further, (n) solvability implies that every element of P_n corresponds to an element in a Witt group. If the knot is $(n.5)$ solvable, then each of these elements in the Witt group is trivial.

They have proven that a knot being (1.5) solvable implies that the Casson-Gordon invariants for the knot vanish, though T. Kim [6] has shown that the converse is not true. Cochrain, Orr and Teichner also have examples of a knots that generate an infinite family of knots that are (2) solvable but not (2.5) solvable. They also know that there are knots that are (n) solvable but not $(n.5)$ solvable for all n.

3 The Future

Clearly our current knowledge of the structure of the group $\mathcal{C}_1$ is far from complete. It seems likely that there are many knots that are slice. It also seems likely that there is no torsion other than 2-torsion in $\mathcal{C}_1$, since the only examples we have of torsion are knots that are concordance order 2, and since we have a theorem of Livingston and Naik eliminating many knots of algebraic order four from being concordance order four. We also know of many families of knots that are infinite order in $\mathcal{C}_1$.

The tools that we have available today for the investigation of these questions have yielded many answers, but they are not the end of the story. There are several examples of knots that are not slice but whose Casson-Gordon invariant vanishes; and there is no reason to think that knots that are (n) and $(n.5)$ solvable for all n are necessarily slice. My hope for the future is that more powerful tools are found, if we wish to understand the knot concordance group.

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